Soliton solutions for quasilinear Schrödinger equations involving supercritical exponent in $\mathbb{R}^N$

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Abstract

We study the existence of positive solutions to the quasilinear elliptic problem

$$-\epsilon\Delta u + V(x)u - \epsilon k(\Delta(|u|^2))u = g(u), \quad u > 0, x \in \mathbb{R}^N,$$

where g has superlinear growth at infinity without any restriction from above on its growth. Mountain pass in a suitable Orlicz space is employed to establish this result. These equations contain strongly singular nonlinearities which include derivatives of the second order which make the situation more complicated. Such equations arise when one seeks for standing wave solutions for the corresponding quasilinear Schrödinger equations. Schrödinger equations of this type have been studied as models of several physical phenomena. The nonlinearity here corresponds to the superfluid film equation in plasma physics.

Key words: Mountain pass, superlinear, standing waves, , quasilinear Schrödinger equations.
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1 Introduction

We are concerned with the existence of positive solutions for quasilinear elliptic equations in the entire space,

$$-\epsilon\Delta u + V(x)u - \epsilon k(\Delta(|u|^2))u = g(u), \quad u > 0, x \in \mathbb{R}^N,$$

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where ϵ is a positive parameter, $V : \mathbb{R}^N \to [0,\infty)$ and $g : [0,\infty) \to [0,\infty)$ are nonnegative continuous functions. Solutions of this equation are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\epsilon\Delta z + W(x)z - l(|z|^2)z - k\epsilon\Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, N > 2, \tag{1}$$

where $W(x)$ is a given potential, k is a real constant and l and h are real functions. Quasilinear equations of the form (1) have been established in several areas of physics corresponding to various types of h. The superfluid film equation in plasma physics has this structure for $h(s) = s$, (Kurihura in [8]). In the case $h(s) = (1+s)^{1/2}$, equation (1) models the self-channeling of a high-power ultra short laser in matter, see [21]. Equation (1) also appears in fluid mechanics [8,9], in the theory of Heidelberg ferromagnetism and magnus [10], in dissipative quantum mechanics [7] and in condensed matter theory [14]. We consider the case $h(s) = s$ and $k > 0$. Setting $z(t,x) = \exp(-iFt)u(x)$ one obtains a corresponding equation of elliptic type which has the formal variational structure:

$$-\epsilon\Delta u + V(x)u - \epsilon k(\Delta(|u|^2))u = g(u), \quad u > 0, x \in \mathbb{R}^N, \tag{2}$$

where $V(x) = W(x) - F$ is the new potential function and g is the new nonlinearity.

Note that, for the case $g(u) = |u|^{p-1}u$ with $N \geq 3$, $p+1 = 22^* = \frac{4N}{N-2}$ behaves like a critical exponent for the above equation [13, Remark 3.13]. For the subcritical case $p+1 < 22^*$ the existence of solutions for problem (2) was studied in [10, 11, 12, 14, 15, 16] and it was left open for the critical exponent case $p+1 = 22^*$ [13; Remark 3.13]. The author in [16], proved the existence of solutions for $p+1 = 22^*$ whenever the potential function $V(x)$ satisfies some geometry conditions. It the present paper, we will extend this result for the supercritical case. It is well-known that for the semilinear case ($k = 0$),

$$-\epsilon\Delta u + V(x)u = g(u), \quad u > 0, x \in \mathbb{R}^N, \tag{P}$$

$p+1 = 2^*$ is the critical exponent when $N \geq 3$. In terms of critical growth, the case $N = 2$ is quite different than $N \geq 3$. We divide, these studies in three cases for problem (P):

- *Subcritical growth:* $\lim_{t\to+\infty} \frac{|g(t)|}{|t|^{2^*}} = 0$, if $N \geq 3$; and $\lim_{t\to+\infty} \frac{|g(t)|}{\exp(\beta t^2)} = 0$ for all $\beta > 0$, if $N = 2$.
- *Critical growth:* $\lim_{t\to+\infty} \frac{|g(t)|}{|t|^{2^*}} = L$ with $L > 0$ if $N \geq 3$; and for $N = 2$, there exists $\beta_0 > 0$ such that
$$\lim_{t\to+\infty} \frac{|g(t)|}{\exp(\beta t^2)} = 0 \qquad \forall \beta > \beta_0, \quad \lim_{t\to+\infty} \frac{|g(t)|}{\exp(\beta t^2)} = +\infty \qquad \forall \beta < \beta_0.$$
- *Supercritical growth:* $\lim_{t\to+\infty} \frac{|g(t)|}{|t|^{2^*}} = +\infty$, if $N \geq 3$; and $\lim_{t\to+\infty} \frac{|g(t)|}{\exp(\beta t^2)} = +\infty$ for all $\beta > 0$, if $N = 2$.

Note that the corresponding critical growth for $N=2$ comes from a version of Moser-Trudinger inequality in whole space $\mathbb{R}^2$ (see [6]) as follows,

$$\int_{\mathbb{R}^2} \big(\exp(\beta|u|^2)-1\big)\,dx < +\infty, \qquad \forall u \in H^1(\mathbb{R}^2), \beta > 0.$$

Also, if $\beta < 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \le C$, there exists a constant $C_2 = C_2(C,\beta)$ such that

$$\sup_{|\nabla u|_{L^2(\mathbb{R}^2)}\le 1} \int_{\mathbb{R}^2} \big(\exp(\beta|u|^2)-1\big)\,dx < C_2.$$

There are many results about the existence of solutions for the subcritical, critical and the supercritical exponent case for problem (P)(e.g. [1, 4, 5, 19, 22]).

In the case $k>0$, for the subcritical case, the existence of a nonnegative solution was proved for $N=1$ by Poppenberg, Schmitt and Wang in [18] and for $N\ge 2$ by Liu and Wang in [12]. In [13] Liu and Wang improved these results by using a change of variables and treating the new problem in an Orlicz space. The author in [15], using the idea of the fibrering method, studied this problem in connection with the corresponding eigenvalue problem for the laplacian $-\Delta u = V(x)u$ and proved the existence of multiple solutions for problem (2). It is established in [11], the existence of both one-sign and nodal ground states of soliton type solutions by the Nehari method. They also established some regularity of the positive solutions.
As it was mentioned above, for the case $k>0$ with $g(u)=|u|^{p-1}u$ and $N\ge 3$, $p+1 = 22^* = \frac{4N}{N-2}$ behaves like a critical exponent for problem (2). This is because of the nonlinearity term $-\epsilon k(\Delta(|u|^2))u$. Therefore for problem (2), because of the presence of this nonlinearity term, the above definition of Subcritical , Critical and Supercritical growth changes as follows:

- ***Subcritical growth:*** $\lim_{t\to+\infty}\frac{|g(t)|}{|t|^{22^*}}=0$, *if* $N\ge 3$*; and* $\lim_{t\to+\infty}\frac{|g(t)|}{\exp(\beta t^4)}=0$ *for all* $\beta>0$, *if* $N=2$.

- ***Critical growth:*** $\lim_{t\to+\infty}\frac{|g(t)|}{|t|^{22^*}}=L$ with $L>0$ if $N\ge 3$; and for $N=2$, there exists $\beta_0>0$ such that

$$\lim_{t\to+\infty}\frac{|g(t)|}{\exp(\beta t^4)}=0 \qquad \forall\beta>\beta_0, \quad \lim_{t\to+\infty}\frac{|g(t)|}{\exp(\beta t^4)}=+\infty \qquad \forall\beta<\beta_0.$$

- ***Supercritical growth:*** $\lim_{t\to+\infty}\frac{|g(t)|}{|t|^{22^*}}=+\infty$, if $N\ge 3$; and $\lim_{t\to+\infty}\frac{|g(t)|}{\exp(\beta t^4)}=+\infty$ for all $\beta>0$, if $N=2$.

Here, we shall study problem (2) with $N\ge 2$ and show the existence of positive solutions when the function g has the supercritical growth. Before to state the main result, we fix the hypotheses on the potential function V and the function g. Indeed, we assume that the potential function V is radial, that is $V(x)=V(|x|)$, and satisfies the following conditions:
There exist $0<R_1<r_1<r_2<R_2$ and $\alpha>0$ such that

A1: $V(x) = 0, \qquad \forall x \in \Omega := \left\{x \in \mathbb{R}^N : r_1 < |x| < r_2\right\},$

A2: $V(x) \geq \alpha, \qquad \forall x \in \Lambda^c,$

where $\Lambda = \left\{x \in \mathbb{R}^N : R_1 < |x| < R_2\right\}$. Also, we assume g is continuous and verifies the following conditions,

H1: $\lim_{t\to+\infty} \frac{g(t)}{t} = +\infty.$

H2: The Ambrosetti-Rabinowitz growth condition: There exists $\theta > 2$ such that

$$0 \leq \theta G(t) = \theta \int_0^t g(s)\, ds \leq tg(t), \qquad t \in \mathbb{R}.$$

H3: $\frac{g(t)}{t}$ is non-decreasing with respect to t, for $t > 0$.

H4: $\lim_{t\to 0} \frac{g(t)}{t} = 0.$

Here is our main Theorem.

Theorem 1.1. *Assume Conditions $H1 - H4, A1$ and $A2$. Then, there exists $\epsilon_0 > 0$, such that for all $\epsilon \in (0, \epsilon_0)$ problem (2) has a nonnegative solution $u_\epsilon \in H_r^1(\mathbb{R}^N)$ with $u_\epsilon^2 \in H_r^1(\mathbb{R}^N)$ and*

$$u_\epsilon(x) \longrightarrow 0 \quad as \quad |x| \longrightarrow +\infty.$$

This paper is organized as follows. In Section 2, we reformulate this problem in an appropriate Orlicz space. In Section 3, we prove the existence of a solution for a special deformation of problem (2). Theorem 1.1 is proved in Section 4.

2 Reformulation of the problem and preliminaries

Denote by $H_r^1(\mathbb{R}^N)$ the space of radially symmetric functions in

$$H^{1,2}(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \bigtriangledown u \in L^2(\mathbb{R}^N)\right\},$$

and by $C_{0,r}^\infty(\mathbb{R}^N)$ the space of radially symmetric functions in $C_0^\infty(\mathbb{R}^N)$. Without loss of generality, one can assume $k = 1$ in problem (2). We formally formulate problem (2) in a variational structure as follows

$$J_\epsilon(u) = \frac{\epsilon}{2} \int_{\mathbb{R}^N} (1 + u^2)| \bigtriangledown u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} G(u)dx.$$

on the space

$$X = \{u \in H_r^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\},$$

which is equipped with the following norm,

$$\|u\|_X = \left\{\int_{\mathbb{R}^N} | \bigtriangledown u|^2 dx + \int_{\mathbb{R}^N} V(x)u^2 dx\right\}^{\frac{1}{2}}.$$

Liu and Wang in [13] for the subcritical case, by making a change of variables treated this problem in an Orlicz space. Following their work, we consider this problem for the supercritical exponent case in the same Orlicz space. To convince the reader we briefly recall some of their notations and results that are useful in the sequel.

First, we make a change of variables as follows,

$$dv = \sqrt{1+u^2}du, \quad v = h(u) = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2}).$$

Since h is strictly monotone it has a well-defined inverse function: $u = f(v)$. Note that

$$h(u) \sim \begin{cases} u, & |u| \ll 1 \\ \frac{1}{2}u|u|, & |u| \gg 1, \quad h'(u) = \sqrt{1+u^2}, \end{cases}$$

and

$$f(v) \sim \begin{cases} v & |v| \ll 1 \\ \sqrt{\frac{2}{|v|}}v, & |v| \gg 1, \quad f'(v) = \frac{1}{h'(u)} = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+f^2(v)}}. \end{cases}$$

Also, for some $C_0 > 0$ it holds

$$L(v) := f(v)^2 \sim \begin{cases} v^2 & |v| \ll 1, \\ 2|v| & |v| \gg 1, \quad L(2v) \le C_0 L(v), \end{cases}$$

$L(v)$ is convex, $L'(v) = \frac{2f(v)}{\sqrt{1+f(v)^2}}$, $L''(v) = \frac{2}{(1+f(v)^2)^2} > 0$.

Using this change of variable, we can rewrite the functional $J_\epsilon(u)$ as

$$\bar{J}_\epsilon(v) = \frac{\epsilon}{2}\int_{\mathbb{R}^N} |\bigtriangledown v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f(v)^2 dx - \int_{\mathbb{R}^N} G(f(v))dx.$$

$\bar{J}_\epsilon$ is defined on the space

$$H_L^1(\mathbb{R}^N) = \{v | v(x) = v(|x|), \int_{\mathbb{R}^N} |\bigtriangledown v|^2 dx < \infty, \int_{\mathbb{R}^N} V(x)L(v)dx < \infty\}.$$

We introduced the Orlicz space (e.g.[20])

$$E_L(\mathbb{R}^N) = \{v | \int_{\mathbb{R}^N} V(x)L(v)dx < \infty\},$$

equipped with the norm

$$|v|_{E_L(\mathbb{R}^N)} = \inf_{\zeta>0} \zeta(1 + \int_{\mathbb{R}^N}(V(x)L(\zeta^{-1}v(x))dx),$$

and define the norm of $H_L^1(\mathbb{R}^N)$ by

$$\|v\|_{H_L^1(\mathbb{R}^N)} = |\bigtriangledown v|_{L^2(\mathbb{R}^N)} + |v|_{E_L(\mathbb{R}^N)}.$$

Here are some related facts. See Propositions (2.1) and (2.2) in [13] for the proof.

Proposition 2.1. *(i) $E_L(\mathbb{R}^N)$ is a Banach space.*

(ii) If $v_n \longrightarrow v$ in $E_L(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} V(x)|L(v_n)-L(v)|dx \longrightarrow 0$ and $\int_{\mathbb{R}^N} V(x)|f(v_n)-f(v)|^2dx \longrightarrow 0$.

(iii) If $v_n \longrightarrow v$ a.e. and $\int_{\mathbb{R}^N} V(x)L(v_n)dx \longrightarrow \int_{\mathbb{R}^N} V(x)L(v)dx$, then $v_n \longrightarrow v$ in $E_L(\mathbb{R}^N)$.

(iv) The dual space $E_L^*(\mathbb{R}^N) = L^\infty \cap L_V^2 = \{w|w \in L^\infty, \int_{\mathbb{R}^N} V(x)w^2dx < \infty\}$.

(v) If $v \in E_L(\mathbb{R}^N)$, then $w = L'(v) = 2f(v)f'(v) \in E_L^*(\mathbb{R}^N)$, and $|w|_{E_L^*} = \sup_{|\phi|_{E_L(\mathbb{R}^N)}\leq 1}(w,\phi) \leq C_1(1+\int_{\mathbb{R}^N} V(x)L(v)dx)$, where C_1 is a constant independent of v.

(vi) For $N > 2$ the map:$v \longrightarrow f(v)$ from $H_L^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is continuous for $2 \leq q \leq 22^*$ and is compact for $2 < q < 22^*$. Also, for $N = 2$, this map is compact for $q > 2$.

Hence forth, $\int, H^1, H_r^1, H_L^1, E_L, L^t, |\cdot|_L$ and $\|\cdot\|$ stand for $\int_{\mathbb{R}^N}$, $H^{1,2}(\mathbb{R}^N)$, $H_r^1(\mathbb{R}^N)$, $H_L^1(\mathbb{R}^N)$, $E_L(\mathbb{R}^N)$, $L^t(\mathbb{R}^N)$, $|\cdot|_{E_L(\mathbb{R}^N)}$ and $\|\cdot\|_{H_L^1(\mathbb{R}^N)}$ respectively. In the following we use C to denote any constant that is independent of the sequences considered.

3 Auxiliary Problem

In this section, we shall show some results needed to prove Theorem 1.1. Indeed, we first consider a special deformation $\bar{H}_\epsilon$ (see (3) in the following) of $\bar{J}_\epsilon$. Then, We show that the functional $\bar{H}_\epsilon$ satisfies all the properties of the Mountain Pass Theorem. Consequently, $\bar{H}_\epsilon$ has a critical point for each $\epsilon > 0$. We shall use this to prove Theorem 1.1 in the next section. In fact, we will see that the functionals $\bar{J}_\epsilon$ and $\bar{H}_\epsilon$ will coincide for the small values of ϵ. This idea was explored by Del Pino and Felmer [5].

To do this, we shall consider constants

$$k > \max\{\frac{\theta}{\theta-2}, 2\}, \qquad (\theta \text{ is introduced in } H2),$$

and a with

$$\frac{g(a)}{a} = \frac{\alpha}{k}, \qquad (\alpha \text{ is introduced in } A2),$$

and functions

$$\bar{g}(s) = \begin{cases} g(s), & s \leq a, \\ (\frac{\alpha}{k})s, & s > a, \end{cases}$$

$$w(x,s) = \chi_\Lambda(x)g(s) + (1-\chi_\Lambda(x))\bar{g}(s),$$

where χ_Λ denotes the characteristic function of the set Λ. Set $W(x,t) = \int_0^t w(x,\zeta)d\zeta$. It is easily seen that the function w satisfies the following conditions,

G1: $0 \le \theta W(x,t) \le w(x,t)t, \quad \forall x \in \Lambda, t \ge 0.$

G2: $0 \le 2W(x,t) \le w(x,t)t \le \frac{1}{k}V(x)t^2, \quad \forall x \in \Lambda^c, t \in \mathbb{R}.$

Also, it is easy to check that w satisfies the condition $H2$. In the sequel, we denote by $G3$, the condition $H2$ with g replaced by w.

Now, we study the existence of solutions for the deformed equation, i.e.

$$-\epsilon\Delta u + V(x)u - \epsilon(\Delta(|u|^2))u = w(x,u), \quad x \in \mathbb{R}^N.$$

which correspond to the critical points of

$$H_\epsilon(u) = \frac{\epsilon}{2}\int(1+u^2)|\bigtriangledown u|^2 + \frac{1}{2}\int V(x)u^2 - \int W(x,u)dx.$$

As in Section (2), we can rewrite the functional $H_\epsilon(u)$ as a new functional $\bar{H}_\epsilon(v)$ with $u = f(v)$ as follows,

$$\bar{H}_\epsilon(v) = \frac{\epsilon}{2}\int|\bigtriangledown v|^2dx + \frac{1}{2}\int V(x)f(v)^2dx - \int W(x,f(v))dx. \tag{3}$$

$\bar{H}_\epsilon(v)$ is defined on the Orlicz space H^1_L. To simplify the writing in this section, we shall assume $\epsilon = 1, H_1 = H$ and $\bar{H}_1 = \bar{H}$.

The following Proposition states some properties of the functional $\bar{H}$.

Proposition 3.1. *(i) $\bar{H}$ is well-defined on H^1_L.*

(ii) $\bar{H}$ is continuous in H^1_L.

(iii) $\bar{H}$ is Gauteaux-differentiable in H^1_L.

Proof. The proof is similar to the proof of Proposition (2.3) in [13] by some obvious changes.□

Here is the main result in this section.

Theorem 3.2. *$\bar{H}$ has a critical point in H^1_L, that is, there exists $0 \ne v \in H^1_L$ such that*

$$\int \bigtriangledown v . \bigtriangledown \phi dx + \int V(x)f(v)f'(v)\phi dx - \int w(x,f(v))f'(v)\phi dx = 0,$$

for every $\phi \in H^1_L$.

We use the Mountain Pass Theorem (see [2], [19]) to prove Theorem 3.2. First, let us define the Mountain Pass value,

$$C_0 := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \bar{H}(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1_L)|\gamma(0) = 0, \bar{H}(\gamma(1)) \le 0, \gamma(1) \ne 0\}.$$

The following Lemmas are crucial for the proof of Theorem 3.2.

Lemma 3.3. *The functional $\bar{H}$ satisfies the Mountain Pass Geometry.*

Proof. We need to show that there exists $0 \neq v \in \; H^1_L$ such that $\bar{H}(v) \leq 0$. Let $e \in C^\infty_{0,r}(\mathbb{R}^N)$ with $e \not\equiv 0$ and $\mathrm{supp}(e) \subset \Omega$. It is easy to see that $H(te) \leq 0$ for the large values of t. Consequently $\bar{H}(v) < 0$ where $v = h(te)$. $\square$

Lemma 3.4. C_0 *is positive.*

Proof. Set

$$S_\rho := \{v \in H^1_L | \int |\bigtriangledown v|^2 dx + \int V(x) f(v)^2 dx = \rho^2\}.$$

It follows from $H4$ that for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$G(t) \leq \frac{\epsilon t^2}{2}, \qquad |t| \leq \delta.$$

Thus

$$\int_\Lambda G(u)\, dx \leq \frac{\epsilon}{2} \int_\Lambda u^2\, dx, \text{ as } \|u\|_X \leq \rho_1 \text{ with } \rho_1 \text{ small enough.}$$

Set $u = f(v)$. It is easy to check that $\|u\|_X \leq \|v\|_{H^1_L(\mathbb{R}^N)}$. Hence, it follows

$$\int_\Lambda G(f(v))\, dx \leq \frac{\epsilon}{2} \int_\Lambda f(v)^2\, dx, \text{ as } \|v\|_{H^1_L(\mathbb{R}^N)} \leq \rho_1 \text{ with } \rho_1 \text{ small enough.}$$

Recalling that

$$\int_\Lambda f(v)^2\, dx \leq C(\int |\bigtriangledown v|^2 dx + \int V(x) f(v)^2 dx),$$

we obtain for each $v \in S_\rho$

$$\int_\Lambda G(f(v))\, dx \leq \frac{C\epsilon}{2}\rho^2, \tag{4}$$

for small values of ρ.
Also, it follows from $(G1)$ and $(G2)$ for each $v \in S_\rho$ with ρ small enough that

$$\begin{aligned} \int W(x, f(v)) dx &= \int_\Lambda W(x, f(v)) dx + \int_{\Lambda^c} W(x, f(v)) dx \\ &\leq \int_\Lambda G(f(v))\, dx + \frac{1}{2k} \int V(x) f(v)^2 dx \\ &\leq \frac{C\epsilon}{2}\rho^2 + \frac{1}{2k}\rho^2 \end{aligned} \tag{5}$$

Considering (4), (5) and the fact that $v \in S_\rho$, we obtain

$$\begin{aligned} \bar{H}(v) &= \frac{1}{2} \int |\bigtriangledown v|^2 dx + \frac{1}{2} \int V(x) f(v)^2 dx - \int W(x, f(v)) dx \\ &\geq \frac{1}{2}\rho^2 - \frac{C\epsilon}{2}\rho^2 - \frac{1}{2k}\rho^2 = (\frac{1}{2} - \frac{1}{2k})\rho^2 - \frac{C\epsilon}{2}\rho^2 \geq \frac{k-1}{4k}\rho^2, \end{aligned}$$

when $0 < \rho \le \rho_0 \ll 1$ for some ρ_0 and ϵ small enough. Hence, for $v \in S_\rho$ with $0 < \rho \le \rho_0$ we have

$$\bar{H}(v) \ge \frac{k-1}{4k}\rho^2. \tag{6}$$

If $\gamma(1) = v$ and $\bar{H}(\gamma(1)) < 0$ then it follows from (6) that

$$\int |\bigtriangledown v|^2 dx + \int V(x) f(v)^2 dx > \rho_0^2,$$

thereby giving

$$\sup_{t\in[0,1]} \bar{H}(\gamma(t)) \ge \sup_{\gamma(t)\in S_{\rho_0}} \bar{H}(\gamma(t)) \ge \frac{k-1}{4k}\rho_0^2.$$

Therefore $C_0 \ge \frac{k-1}{4k}\rho_0^2 > 0.\square$

The Mountain Pass Theorem guaranties the existence of a $(PS)_{C_0}$ sequence $\{v_n\}$, that is, $\bar{H}(v_n) \longrightarrow C_0$ and $\bar{H}'(v_n) \longrightarrow 0$. The following lemma states some properties of this sequence.

Lemma 3.5. *Suppose $\{v_n\}$ is a $(PS)_{C_0}$ sequence. The following statements hold.*

(i) $\{v_n\}$ is bounded in H_L^1.

(ii) For each $\delta > 0$, there exists $R > 4R_2$, (R_2 is introduced in (A1) and (A2)) such that

$$\limsup_{n\to+\infty} \int_{B_R^c} \big(|\bigtriangledown v_n|^2 + V(x) f(v_n)^2\big) dx < \delta.$$

(iii) If v_n converges weakly to v in H_L^1, then

$$\lim_{n\to+\infty} \int w(x, f(v_n)) f(v_n) dx = \int w(x, f(v)) f(v) dx.$$

(iv) If $v_n \ge 0$ converges weakly to v in H_L^1, then for every nonnegative test function $\phi \in H_L^1$ we have

$$\lim_{n\to+\infty} \langle \bar{H}'(v_n), \phi\rangle = \langle \bar{H}'(v), \phi\rangle.$$

Proof. Since $\{v_n\}$ is a $(PS)_{C_0}$ sequence, we have

$$\begin{aligned}\bar{H}(v_n) &= \frac{1}{2}\int |\bigtriangledown v_n|^2 dx + \frac{1}{2}\int V(x) f(v_n)^2 dx - \int W(x, f(v_n)) dx \\ &= C_0 + o(1),\end{aligned} \tag{7}$$

and

$$\begin{aligned}\langle \bar{H}'(v_n), \phi\rangle &= \int \bigtriangledown v_n . \bigtriangledown \phi dx + \int V(x) f(v_n) f'(v_n)\phi dx - \int w(x, f(v_n)) f'(v_n)\phi dx \\ &= o(\|\phi\|)\end{aligned} \tag{8}$$

For part (i), pick $\phi = \frac{f(v_n)}{f'(v_n)} = \sqrt{1+f(v_n)^2} f(v_n)$ as a test function. One can easily deduce that $|\phi|_L \le C|v_n|_L$ and

$$|\bigtriangledown \phi| = (1 + \frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n| \le 2|\bigtriangledown v_n|,$$

which implies $\|\phi\| \leq C\|v_n\|$. Substituting ϕ in (8), gives

$$\begin{aligned}\langle \bar{H}'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle &= \int (1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2 dx + \int V(x) f(v_n)^2 dx \\ &\quad - \int w(x, f(v_n)) f(v_n) dx \\ &= o(\|v_n\|). \end{aligned} \tag{9}$$

It follows from $(G1)$ and $(G2)$ that

$$-\int W(x, f(v_n))dx + \frac{1}{\theta}\int w(x, f(v_n)) f(v_n) dx \geq \frac{1}{k}(\frac{1}{\theta} - \frac{1}{2}) \int V(x) f(v_n)^2 dx \tag{10}$$

Taking into account (7), (9) and (10), we have

$$\begin{aligned} C_0 + o(1) + o(\|v_n\|) =& \bar{H}(v_n) - \frac{1}{\theta}\langle \bar{H}'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle \\ =& \frac{1}{2}\int |\bigtriangledown v_n|^2 dx + \frac{1}{2}\int V(x) f(v_n)^2 dx - \int W(x, f(v_n)) dx \\ & - \frac{1}{\theta}\int (1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2 dx - \frac{1}{\theta}\int V(x) f(v_n)^2 dx \\ & + \frac{1}{\theta}\int w(x, f(v_n)) f(v_n) dx \\ =& \int (\frac{1}{2} - \frac{1}{\theta}(1+\frac{f(v_n)^2}{1+f(v_n)^2}))|\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})\int V(x) f(v_n)^2 dx \\ & + \int (\frac{1}{\theta} w(x, f(v_n)) f(v_n) - W(x, f(v_n))) dx \\ \geq& (\frac{1}{2} - \frac{2}{\theta})\int |\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})(1 - \frac{1}{k})\int V(x) f(v_n)^2 dx. \end{aligned}$$

Since, $\frac{1}{2} - \frac{2}{\theta} > 0$ and $(\frac{1}{2} - \frac{1}{\theta})(1 - \frac{1}{k}) > 0$ it follows from the above that $\int |\bigtriangledown v_n|^2 dx + \int V(x) f(v_n)^2 dx$ is bounded. It proves part (i).

For part (ii), let $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a function satisfying $\eta_R = 0$ on $B_{\frac{R}{2}}$, $\eta_R = 1$ on B_R^c and $|\bigtriangledown \eta_R(x)| \leq \frac{C}{R}$. It follows from part (i) that $\{v_n\}$ is bounded. Hence, from (8) we have

$$\langle \bar{H}'(v_n), \frac{f(v_n)}{f'(v_n)}\eta_R\rangle = o(1),$$

thereby giving

$$\begin{aligned}\int (1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2 \eta_R dx + \int V(x) f(v_n)^2 \eta_R dx \\ + \int \frac{f(v_n)}{f'(v_n)} \bigtriangledown v_n . \bigtriangledown \eta_R dx = \int w(x, f(v_n)) f(v_n) \eta_R dx + o(1).\end{aligned}$$

By $(G2)$, we get

$$w(x, f(v_n)) f(v_n) \leq \frac{V(x)}{k} f(v_n)^2, \quad \forall x \in B^c_{\frac{R}{2}}.$$

Therefore,

$$\begin{aligned}\int(1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2\eta_R dx+\int(1-\frac{1}{k})V(x)f(v_n)^2\eta_R dx\\ \leq \frac{C}{R}\int\frac{|f(v_n)|}{f'(v_n)}|\bigtriangledown v_n|dx+o(1)\\ \leq \frac{C}{R}\int|\bigtriangledown v_n|^2dx+\frac{C}{R}\int(|f(v_n)|^2+|f(v_n)|^4)dx+o(1).\end{aligned}\tag{11}$$

Also, it follows from part (vi) of Proposition 2.1 that $\{f(v_n)\}_n$ is a bounded sequence in $L^2(\mathbb{R}^N)\cap L^4(\mathbb{R}^N)$. Hence, $\int(|f(v_n)|^2+|f(v_n)|^4)dx$ is bounded. Therefore, it follows from (11) that

$$\limsup_{n\to\infty}\int_{B_R^c}\big(|\bigtriangledown v_n|^2dx+V(x)f(v_n)^2\big)dx<\delta,\quad (R>4R_2).$$

It proves part (ii).

For part (iii), note first that from part (ii) of the present Lemma for each $\delta>0$ there exists $R>4R_2$ such that

$$\limsup_{n\to\infty}\int_{B_R^c}\big(|\bigtriangledown v_n|^2+V(x)f(v_n)^2\big)dx<\frac{k\delta}{4}.\tag{12}$$

Since $B_R^c\subseteq\Lambda^c$, it follows from $(G2)$ that

$$w(x,f(v_n))f(v_n)\leq\frac{V(x)}{k}f(v_n)^2\qquad\forall x\in B_R^c$$

which together with (12) imply that

$$\limsup_{n\to\infty}\int_{B_R^c}w(x,f(v_n))f(v_n)dx\leq\frac{\delta}{4},\tag{13}$$

and consequently

$$\int_{B_R^c}w(x,f(v))f(v)dx\leq\frac{\delta}{4}.$$

It follows from (13) and the above inequality that

$$\begin{aligned}\Big|\int w(x,f(v_n))f(v_n)dx-\int w(x,f(v))f(v)dx\Big|\\ \leq\frac{\delta}{2}+\Big|\int_{B_{R_1}}\big[w(x,f(v_n))f(v_n)-w(x,f(v))f(v)\big]dx\Big|\\ +\Big|\int_{B_R\setminus B_{R_1}}\big[w(x,f(v_n))f(v_n)-w(x,f(v))f(v)\big]dx\Big|.\end{aligned}\tag{14}$$

Since $B_{R_1}\subset\Lambda^c$, we have

$$w(x,f(v_n))f(v_n)\leq\frac{V(x)}{k}f(v_n)^2,\quad\forall x\in B_{R_1}$$

Then, by the compact theorem embedding and Lebesgue Theorem, we obtain a subsequence still denoted by $\{v_n\}$, such that

$$\int_{B_{R_1}}w(x,f(v_n))f(v_n)dx\longrightarrow\int_{B_{R_1}}w(x,f(v))f(v)dx.\tag{15}$$

Now, we show that

$$\int_{B_R\setminus\bar{B}_{R_1}} w(x, f(v_n))f(v_n)dx \longrightarrow \int_{B_R\setminus\bar{B}_{R_1}} w(x, f(v))f(v)dx.$$

Since $v_n \rightharpoonup v$ weakly in H^1_L, there exists a constant C such that $\|v_n\| \le C$. Set $u_n = f(v_n)$. An easy computation shows that $\|u_n\|_X \le \|v_n\| \le C$. Using Straus's inequality (see [22]) we have

$$|u_n(x)| \le \frac{2\pi}{|x|^{\frac{1}{2}}}\|u_n\|_X \le \frac{2\pi C}{|x|^{\frac{1}{2}}}, \qquad \forall x \ne 0,$$

from which

$$|u_n(x)| \le \frac{2\pi C}{R_1^{\frac{1}{2}}} := \bar{C}, \qquad \forall x \in B_R\setminus\bar{B}_{R_1}.$$

From this we have

$$|w(x, f(v_n))v_n| = |w(x, u_n)u_n| \le \max_{x\in B_R\setminus\bar{B}_{R_1}, t\in[-\bar{C},\bar{C}]} w(x,t)\bar{C} := \bar{C}_0 \in L^1(B_R\setminus\bar{B}_{R_1}).$$

Then, it follows from the Lebesgue dominated convergence theorem that

$$\int_{B_R\setminus\bar{B}_{R_1}} w(x, f(v_n))f(v_n)dx \longrightarrow \int_{B_R\setminus\bar{B}_{R_1}} w(x, f(v))f(v)dx. \tag{16}$$

Considering (15) and (16), it follows from (14) that

$$\limsup_{n\to\infty}\Big|\int w(x, f(v_n))f(v_n)dx - \int w(x, f(v))f(v)dx\Big| \le \frac{\delta}{2},$$

for every $\delta > 0$. Consequently

$$\int w(x, f(v_n))f(v_n)dx \longrightarrow \int w(x, f(v))f(v)dx,$$

as $n \to \infty$. It proves part (iii).

To prove part (iv), note first that f is increasing and $f(0) = 0$, hence $f(v_n) \ge 0$ and $f(v) \ge 0$. For the second term on the right hand side of (8), we have

$$V(x)f(v_n)f'(v_n)\phi \le V(x)f(v_n)\phi,$$

and since $v_n \rightharpoonup v$ weakly in H^G_1, for the right hand side of the above inequality we have

$$\lim_{n\to\infty}\int V(x)f(v_n)\phi\,dx = \int V(x)f(v)\phi\,dx.$$

Hence by the dominated convergence theorem and the fact that $v_n \to v$ a.e. we obtain

$$\lim_{n\to\infty}\int V(x)f(v_n)f'(v_n)\phi\,dx = \int V(x)f(v)f'(v)\phi\,dx. \tag{17}$$

For the third term on the right hand side of (8), we have

$$w(x, f(v_n))f'(v_n)\phi \le \frac{V(x)}{k} f(v_n)\phi, \qquad \forall x \in \Lambda^c,$$

and similarly by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{\Lambda^c} w(x, f(v_n))f'(v_n)\phi\, dx = \int_{\Lambda^c} w(x, f(v))f'(v)\phi\, dx. \tag{18}$$

Also, by the same argument to prove (16), we obtain

$$\lim_{n\to\infty} \int_{\Lambda} w(x, f(v_n))f'(v_n)\phi\, dx = \int_{\Lambda} w(x, f(v))f'(v)\phi\, dx. \tag{19}$$

It follows from (8) and (17)-(19) that

$$\lim_{n\to+\infty} \langle \bar{H}'(v_n), \phi\rangle = \langle \bar{H}'(v), \phi\rangle.$$

It proves part (iv). $\square$

Lemma 3.6. *If $\{v_n\}$ is a $(PS)_{C_0}$ sequence, then v_n converges to $v \in H^1_L$. Consequently $\bar{H}(v) = \lim_{n\to+\infty} \bar{H}(v_n)$ and $\bar{H}'(v) = 0$.*

Proof. It follows from part (i) of Lemma 3.5 that v_n is a bounded sequence in H^1_L. Hence, there exists $v \in H^1_L$ such that, up to a subsequence, $v_n \rightharpoonup v$ weakly in H^1_L and $v_n \to v$ a.e. in $\mathbb{R}^N$. Since we may replace v_n by $|v_n|$, we assume $v_n \ge 0$ and $v \ge 0$. Since, $\{v_n\}$ is a $(PS)_{C_0}$ sequence we have

$$\begin{aligned} o(\|v_n\|) &= \langle \bar{H}'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle \\ &= \int (1 + \frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2 dx + \int V(x) f(v_n)^2 dx \quad - \int w(x, f(v_n))f(v_n) dx \end{aligned} \tag{20}$$

and

$$o(\|v\|) = \langle \bar{H}'(v_n), \frac{f(v)}{f'(v)}\rangle. \tag{21}$$

It follows from part (iv) of Lemma 3.5 and (21) that

$$\begin{aligned} \langle \bar{H}'(v_n), \frac{f(v)}{f'(v)}\rangle =& \langle \bar{H}'(v), \frac{f(v)}{f'(v)}\rangle + o(\|v\|) \\ =& \int (1 + \frac{f(v)^2}{1+f(v)^2})|\bigtriangledown v|^2 dx + \int V(x) f(v)^2 dx \\ &- \int w(x, f(v))f(v) dx + o(\|v\|) \end{aligned} \tag{22}$$

In this step, we show that

$$\int \frac{f(v)^2|\bigtriangledown v|^2}{1+f(v)^2}\, dx \le \liminf_{n\to\infty} \int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\, dx.$$

Set $u_n = f(v_n)$ and $u = f(v)$. A direct computation shows that

$$\int | \bigtriangledown u_n^2|^2 \, dx = 4 \int \frac{f(v_n)^2 | \bigtriangledown v_n|^2}{1 + f(v_n)^2} \, dx \leq 4\|v_n\|^2.$$

Also, from part (vi) of Proposition 2.1 we have

$$\int u_n^4 \, dx = \int f(v_n)^4 \, dx \leq C\|v_n\|^4.$$

Set $w_n = u_n^2$. It follows from the above that $\{w_n\}_n$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Hence, up to a subsequence $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$ and $w_n \to w$ a.e. in $\mathbb{R}^N$. It follows $w = u^2$. Also, by the lower semi continuity of the norm in $H^1(\mathbb{R}^N)$, we obtain

$$\int | \bigtriangledown w|^2 \, dx \leq \liminf_{n\to\infty} \int | \bigtriangledown w_n|^2 \, dx.$$

Plug $w_n = u_n^2$ and $w = u^2$ in this inequality to get

$$\int | \bigtriangledown u^2|^2 \, dx \leq \liminf_{n\to\infty} \int | \bigtriangledown u_n^2|^2 \, dx.$$

Substituting $u_n = f(v_n)$ and $u = f(v)$ in the above inequality gives

$$\int \frac{f(v)^2 | \bigtriangledown v|^2}{1 + f(v)^2} \, dx \leq \liminf_{n\to\infty} \int \frac{f(v_n)^2 | \bigtriangledown v_n|^2}{1 + f(v_n)^2} \, dx. \tag{23}$$

Also, lower semi continuity and Fatou's Lemma imply

$$\int | \bigtriangledown v|^2 dx \leq \liminf_{n\to\infty} \int | \bigtriangledown v_n|^2 dx, \tag{24}$$

$$\int V(x)L(v)dx \leq \liminf_{n\to\infty} \int V(x)L(v_n)dx. \tag{25}$$

Up to a subsequence one can assume

$$\liminf_{n\to\infty} \int | \bigtriangledown v_n|^2 dx = \lim_{n\to\infty} \int | \bigtriangledown v_n|^2 dx \tag{26}$$

$$\liminf_{n\to\infty} \int V(x)L(v_n)dx = \lim_{n\to\infty} \int V(x)L(v_n)dx. \tag{27}$$

$$\liminf_{n\to\infty} \int \frac{f(v_n)^2 | \bigtriangledown v_n|^2}{1 + f(v_n)^2} \, dx = \lim_{n\to\infty} \int \frac{f(v_n)^2 | \bigtriangledown v_n|^2}{1 + f(v_n)^2} \, dx. \tag{28}$$

It follows from (23)-(28) that there exist nonnegative numbers δ_1, δ_2 and δ_3 such that

$$\lim_{n\to\infty} \int | \bigtriangledown v_n|^2 dx = \int | \bigtriangledown v|^2 dx + \delta_1 \tag{29}$$

$$\lim_{n\to\infty} \int V(x)L(v_n)dx = \int V(x)L(v)dx + \delta_2. \tag{30}$$

$$\lim_{n\to\infty} \int \frac{f(v_n)^2 | \bigtriangledown v_n|^2}{1 + f(v_n)^2} \, dx = \int \frac{f(v)^2 | \bigtriangledown v|^2}{1 + f(v)^2} \, dx + \delta_3. \tag{31}$$

Now, we show that $\delta_1 = \delta_2 = \delta_3 = 0$. It follows from part (iii) of Lemma 3.5 that

$$\int w(x, f(v_n))f(v_n)dx \longrightarrow \int w(x, f(v))f(v)dx.$$

which together with (20) and (22) imply

$$\begin{aligned}\lim_{n\to\infty} \Big\{ \int (1 + \frac{f(v_n)^2}{1+f(v_n)^2})| \bigtriangledown v_n|^2 dx \\ + \int V(x)f(v_n)^2 dx\Big\} &= \lim_{n\to\infty} \int w(x, f(v_n))f(v_n)dx \\ &= \int w(x, f(v))f(v)dx \\ &= \int (1 + \frac{f(v)^2}{1+f(v)^2})| \bigtriangledown v|^2 dx + \int V(x)f(v)^2 dx\end{aligned}$$

Taking into account (29), (30) and (31) the above limit implies $\delta_1 = \delta_2 = \delta_3 = 0$. Therefore, it follows from (29) and (30) that

$$\int | \bigtriangledown v|^2 dx = \lim_{n\to\infty} \int | \bigtriangledown v_n|^2 dx$$

$$\int V(x)L(v)dx = \lim_{n\to\infty} \int V(x)L(v_n)dx.$$

By Proposition 2.1, $v_n \longrightarrow v$ in E_L and we have $\bigtriangledown v_n \longrightarrow \bigtriangledown v$ in L^2. Hence $v_n \longrightarrow v$ in H_L^1. $\square$

Proof of Theorem 3.2. The proof is a direct consequence of Lemmas 3.3, 3.4 and 3.5. $\square$

4 Proof of Theorem 1.1

To prove Theorem 1.1, note first that every critical point of the functional $\bar{J}_\epsilon$ corresponds to a weak solution of problem (2). Thus, we need to find a critical point for the functional $\bar{J}_\epsilon$. To do this, we shall show that the functionals $\bar{J}_\epsilon$ and $\bar{H}_\epsilon$ will coincide for the small values of ϵ. Hence, every critical point of $\bar{H}_\epsilon$ will be a critical point of $\bar{J}_\epsilon$. Also, it follows from Theorem 3.2 that $\bar{H}_\epsilon$ has a nontrivial critical point for every $\epsilon > 0$.

Without loss of generality, we may assume ϵ^2 instead of ϵ in the functionals $\bar{H}_\epsilon$ and $\bar{J}_\epsilon$, i.e.

$$\bar{H}_\epsilon(v) = \frac{\epsilon^2}{2} \int | \bigtriangledown v|^2 + \frac{1}{2} \int V(x)f(v)^2 dx - \int W(x, f(v))dx,$$

and

$$\bar{J}_\epsilon(v) = \frac{\epsilon^2}{2} \int_{\mathbb{R}^N} | \bigtriangledown v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f(v)^2 dx - \int_{\mathbb{R}^N} G(f(v))dx.$$

It follows from Theorem 3.2 that there exists a critical point $v_\epsilon \in H_L^1$ of $\bar{H}_\epsilon(v)$ for each $\epsilon > 0$. Set $u_\epsilon = f(v_\epsilon)$.

The following Lemmas are crucial for the proof of Theorem 1.1.

Lemma 4.1. *The sequence $\{u_\epsilon\}_{\epsilon>0}$ is strongly convergent to 0 when $\epsilon \longrightarrow 0$, in $H^1(\mathbb{R}^N)$, i.e.*

$$\|u_\epsilon\|_{H^1} \longrightarrow 0 \quad as \quad \epsilon \longrightarrow 0.$$

Proof. Let $0 \not\equiv \phi \in C^\infty_{0,r}(\mathbb{R}^N)$ be a non-negative function with $\mathrm{supp}(\phi) \subset \Omega$ and $H_1(\phi) \leq 0$. Set $\gamma_1(t) := h(t\phi)$. Hence, we have

$$\bar{H}_\epsilon(\gamma_1(1)) = \bar{H}_\epsilon(h(\phi)) = H_\epsilon(\phi) \leq H_1(\phi) \leq 0.$$

It follows from the definition of the Mountain Pass value that

$$\bar{H}_\epsilon(v_\epsilon) = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \bar{H}_\epsilon(\gamma(t)) \leq \sup_{t\in[0,1]} \bar{H}_\epsilon(\gamma_1(t)) = \sup_{t\in[0,1]} \bar{H}_\epsilon(h(t\phi)) = \sup_{t\in[0,1]} H_\epsilon(t\phi).$$

Therefore, we obtain

$$\begin{aligned}
\bar{H}_\epsilon(v_\epsilon) &\leq \sup_{t\in[0,1]} H_\epsilon(t\phi) \\
&= \sup_{t\in[0,1]} \frac{\epsilon^2 t^2}{2} \int |\bigtriangledown \phi|^2 + \frac{\epsilon^2 t^4}{2} \int |\phi|^2 |\bigtriangledown \phi|^2 - \int G(t\phi)dx \\
&\leq \sup_{t\in[0,1]} \frac{\epsilon^2 t^2}{2} \int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx - \int G(t\phi)dx \\
&= \frac{\epsilon^2 t_\epsilon^2}{2} \int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx - \int G(t_\epsilon\phi)dx
\end{aligned} \tag{32}$$

for some $0 < t_\epsilon < 1$. On the other hand we have

$$\epsilon^2 \int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx = \int \frac{g(t_\epsilon\phi)\phi}{t_\epsilon} dx.$$

Choosing $\Omega_1 \subseteq \Omega$ such that $\phi(x) \geq \phi_0 > 0, \forall x \in \Omega_0$, it follows

$$\epsilon^2 \int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx \geq \int_{\Omega_0} \frac{g(t_\epsilon\phi)\phi}{t_\epsilon} dx \geq \phi_0^2 \int_{\Omega_0} \frac{g(t_\epsilon\phi)}{t_\epsilon\phi} dx.$$

Thus, from the above inequalities and Conditions $H1-H3$ we obtain $t_\epsilon \to 0$ as $\epsilon \to 0$. Now, as in the proof of part (i) of Lemma 3.5 we obtain

$$\begin{aligned}
\bar{H}_\epsilon(v_\epsilon) &= \bar{H}_\epsilon(v_\epsilon) - \frac{1}{\theta}\langle \bar{H}'(v_\epsilon), v_\epsilon\rangle \\
&\geq \epsilon^2(\frac{1}{2} - \frac{2}{\theta}) \int |\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})(1 - \frac{1}{k}) \int V(x) f(v_n)^2 dx.
\end{aligned} \tag{33}$$

Combining (32) and (33), we get

$$\begin{aligned}
\epsilon^2(\frac{1}{2} - \frac{2}{\theta}) \int |\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})(1 - \frac{1}{k}) \int V(x)|f(v_n)|^2 dx \quad &\leq \quad \frac{\epsilon^2 t_\epsilon^2}{2} \int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx \\
&\quad - \int G(t_\epsilon\phi)dx \\
&\leq \quad \frac{\epsilon^2 t_\epsilon^2}{2} \int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx.
\end{aligned}$$

Therefore

$$(\frac{1}{2}-\frac{2}{\theta})\int|\bigtriangledown v_n|^2dx+(\frac{1}{2}-\frac{1}{\theta})(1-\frac{1}{k})\int V(x)f(v_n)^2dx\leq\frac{t_\epsilon^2}{2}\int(1+|\phi|^2)|\bigtriangledown\phi|^2dx. \tag{34}$$

Hence, substituting $u_\epsilon=f(v_\epsilon)$ in (34) implies

$$\int(1+|u_\epsilon|^2)|\bigtriangledown u_\epsilon|^2dx+\int V(x)|u_\epsilon|^2dx\leq\frac{t_\epsilon^2}{2}\int(1+|\phi|^2)|\bigtriangledown\phi|^2dx.$$

Therefore

$$\|u_\epsilon\|_{H^1}\longrightarrow 0\quad\text{as}\quad\epsilon\longrightarrow 0.$$

□

Lemma 4.2. *For every compact set $Q\subset\mathbb{R}^N$ such that $0\notin Q$, $\|u_\epsilon\|_{L^\infty(Q)}\longrightarrow 0$ as $\epsilon\longrightarrow 0$.*

Proof. For each $\epsilon>0$, it follows from Straus's inequality that

$$0\leq u_\epsilon(x)\leq\frac{2\pi}{|x|^{\frac{1}{2}}}\|u_\epsilon\|_{H^1(\mathbb{R}^N)}\quad\forall x\neq 0,$$

which together with the result of Lemma 4.1 obviously means

$$\|u_\epsilon\|_{L^\infty(Q)}\longrightarrow 0\quad\text{as}\quad\epsilon\longrightarrow 0.$$

□

Proof of Theorem 1.1. By Lemma 4.2 we have

$$M_\epsilon:=\max_{x\in\bar{\Lambda}}f(v_\epsilon)\longrightarrow 0\quad\text{as}\quad\epsilon\longrightarrow 0. \tag{35}$$

From (35) there exists $\epsilon_0>0$ such that $\max_{x\in\bar{\Lambda}}f(v_\epsilon)<a$ for every $0<\epsilon<\epsilon_0$. Using the test function $\phi=\frac{(f(v_\epsilon)-a)_+}{f'(v_\epsilon)}$, we get

$$\begin{aligned}0=\langle\bar{H}'_\epsilon(v_\epsilon),\phi\rangle=\int_F\epsilon^2(1+\frac{f(v_\epsilon)^2}{1+f(v_\epsilon)^2})|\bigtriangledown v_\epsilon|^2 &+\int_{\mathbb{R}^N\backslash\bar{\Lambda}}V(x)f(v_\epsilon)(f(v_\epsilon)-a)_+dx\\ &-\int_{\mathbb{R}^N\backslash\bar{\Lambda}}w(x,f(v_\epsilon))(f(v_\epsilon)-a)_+dx\end{aligned}$$

where $F=(\mathbb{R}^N\backslash\bar{\Lambda})\cap\{x|f(v_\epsilon)\geq a\}$. From $(G2)$, we have

$$V(x)f(v_\epsilon)(f(v_\epsilon)-a)_+-w(x,f(v_\epsilon))(f(v_\epsilon)-a)_+\geq 0,\quad\forall x\in\Lambda^c.$$

Thus,

$$\epsilon^2\int_F(1+\frac{f(v_\epsilon)^2}{1+f(v_\epsilon)^2})|\bigtriangledown v_\epsilon|^2dx=0,$$

from which we obtain

$$f(v_\epsilon) \leq a, \quad \forall x \in \mathbb{R}^N \backslash \bar{\Lambda}.$$

Therefore

$$w(x, f(v_\epsilon)) = g(f(v_\epsilon)), \quad \forall x \in \mathbb{R}^N \backslash \bar{\Lambda},$$

and we conclude that

$$\epsilon^2 \int \nabla v_\epsilon . \nabla \xi dx + \int V(x) f(v_\epsilon) f'(v_\epsilon) \xi dx = \int g(f(v_\epsilon)) f'(v_\epsilon) \xi dx$$

for every $\xi \in H^1_L$ and $\epsilon \in (0, \epsilon_0)$. Therefore, $\bar{J}_\epsilon(v)$ has a critical point v_ϵ in H^1_L for every $\epsilon \in (0, \epsilon_0)$. □

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